Stabilization of metastable states

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An exact solution is obtained for a particle moving in a piecewise square nonsymmetric potential of fluctuating height. It turns out that the population of a metastable state increases in the presence of fluctuations similar to a previously found effect due to an external periodic field.

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As is well known [1], a simple pendulum is stable (metastable) in the vertically downward (upward) position. One can, however, stabilize a metastable position by applying high-frequency parametric oscillations to a pendulum (the "Kapitza pendulum"). As was shown both numerically [2] and analytically [3], the "dynamic stabilization" of a pendulum can be also performed by adding an additive, and not multiplicative, periodic field. This raises the question of whether this phenomenon is general, and what the possibilities for stabilization of metastable states are. Due to a large number of examples of metastable states in science and their applications, this problem may be of practical use. Recently we considered [4] the impact of an external periodic field on the populations of two asymmetric energy levels by using a perturbation expansion in the field amplitude to the master equation for discrete, and to the Fokker-Planck equation for space-extended, systems. It turns out that an external field is able to increase the population of the "shallow" state and under some conditions, even to transform it into the "deep" state. In other words, the less stable state may become "more stable" in the presence of an external field ("stabilization" of the metastable state). In both the indicated examples the stabilization of a metastable state was performed through the use of an external periodic field, and the question arises whether this way is unique, or are there other ways to reach such stabilization. As we know from the very popular theory in recent years, the phenomenon of stochastic resonance (SR) [5], an external frequency needed for SR might be introduced to a system either by an external field or by the fluctuation of some of the parameters of a system. Our aim here is to check whether one can stabilize a metastable state by the latter method.

To this end we consider the simplest model of a metastable state that allows an exact analytical solution. As was shown in the last 10–20 years many fundamental properties of a particle moving in a nonlinear potential are generic, i.e., in particular, they are not too sensitive to details of the potential. Therefore, it is worthwhile to consider the simplest potential that allows an analytic solution in addition to numerical simulations for more complicated potentials. Our model (Fig. 1) involves a particle moving under the influence of white noise of strength $2D$ in the piecewise double-well potential restricted by reflecting walls. The potential barriers U_1 and U_2 are different for the right (stable) and the left (metastable) states.

The Fokker-Planck equation for the probability function $P(x,t)$ for the position x of a diffusive particle at the time t is

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}P+D\frac{\partial P}{\partial x}\right]\equiv-\frac{\partial J}{\partial x}, \tag{1}$$

where the probability current J is defined in Eq. (1), and D is the diffusion coefficient.

For the potential $U(x)$ shown in Fig. 1, $\partial U/\partial x=0$, and Eq. (1) reduces to a simple diffusion equation. The potential barriers enter the matching conditions, namely, one has to solve Eq. (1) into each region of $U(x)=\text{const}$, and then to ensure the continuity of P and J on the boundaries of these regions. Continuity of probability current J, which according to Eq. (1) can be written as $J=-De^{-U/D}d/dx(e^{U/D}P)$, means that at points z_i of the jumps of potentials,

$$e^{U_i(z+0)/D}P(z_i+0,t)=e^{U_i(z-0)/D}P(z_i-0,t), \tag{2}$$

$$\frac{\partial P(z_i+0,t)}{\partial x}=\frac{\partial P(z_i-0,t)}{\partial x}, \tag{3}$$

where for $i=1,2$, $z_{1,2}=\pm a$.

The matching conditions (2) and (3) have to be complemented by reflected boundary conditions at the positions $z_j=\pm L$ of the walls,

$$\frac{\partial P(z_j,t)}{\partial x}=0. \tag{4}$$

Here and in the following we are interested only in the stationary solution of Eq. (1) ($\partial P/\partial t=0$). It is easy to check, that the normalized equilibrium probability distribution function $P(x)$ has the following form in each of three regions shown in Fig. 1 [6]:

$$P_1=\frac{\exp(U_1/D)}{2a+(L-a)[\exp(U_1/D)+\exp(U_2/D)]},$$

$$P_2=\frac{1}{2a+(L-a)[\exp(U_1/D)+\exp(U_2/D)]},$$

$$P_3=\frac{\exp(U_2/D)}{2a+(L-a)[\exp(U_1/D)+\exp(U_2/D)]}. \tag{5}$$

We assume that two wells have equal widths, therefore the equilibrium populations of the left (metastable) state is smaller than that of the right (stable) state in the ratio $\exp(U_2-U_1)/D$.

Let the potential barrier undergo the dichotomous fluctuations of the height Δ with flipping rate α, i.e., the barrier for

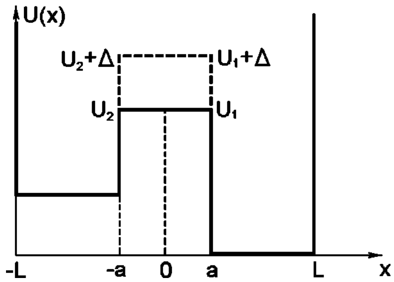

FIG. 1. Square-well potential $U(x)$ of width $2a$ and heights U_1 and U_2 subject to dichotomous fluctuations of height Δ with reflecting boundaries at $x=\pm L$.

the right (left) well changes randomly between heights $U_1, U_1+\Delta(U_2, U_2+\Delta)$. The purpose of our calculation is to check whether the dichotomous fluctuations of the potential barrier are able (like an external periodic field) to increase the population of the metastable well, i.e., to stabilize a metastable state. In this case one has to consider two probability functions, $P_\pm(x)$, where $P_+(P_-)$ defines the probabilities to be at position x when the potential is $U_i+\Delta(U_i)$. Two functions, P_+ and P_-, are related by the following equations [7]:

$$D\frac{d^2P_\pm}{dx^2}+\alpha(P_\mp-P_\pm)=0, \tag{6}$$

which is the simple generalization of the stationary form of Eq. (1) where the transitions between $\pm$ states are taken into account. Differential equations for each of these functions immediately follow from Eq. (6),

$$\frac{d^4P\pm}{dx^4}=\beta^2\frac{d^2P\pm}{dx^2}, \quad \beta^2\equiv\frac{2\alpha}{D}. \tag{7}$$

Solutions of Eq. (7) that satisfy Eq. (6) have the following form:

$$P_\pm^{(1)}(x)=a_1x+a_2\pm a_3\sinh(\beta x)\pm a_4\cosh(\beta x). \tag{8}$$

The population n_1 of the region $[a,L]$ is defined as

$$n_1=\int_a^L[P_+(x)+P_-(x)]dx=a_1(L-a)^2+2a_2(L-A). \tag{9}$$

We added an additional superscript (1) in Eq. (8) showing thereby that it relates to the region $[a,L]$ in Fig. 1. In regions $[-a,a]$ and $[-L,-a]$ the general solutions of Eq. (6) have the same functional form with different constants,

$$P_\pm^{(2)}(x)=b_1x+b_2\pm b_3\sinh(\beta x)\pm b_4\cosh(\beta x), \tag{10}$$

$$P_\pm^{(3)}(x)=c_1x+c_2\pm c_3\sinh(\beta x)\pm c_4\cosh(\beta x). \tag{11}$$

The coefficients in Eqs. (8)–(11) have to be found from the boundary conditions (4) and matching conditions (2)–(3). From the boundary condition at the walls, one gets

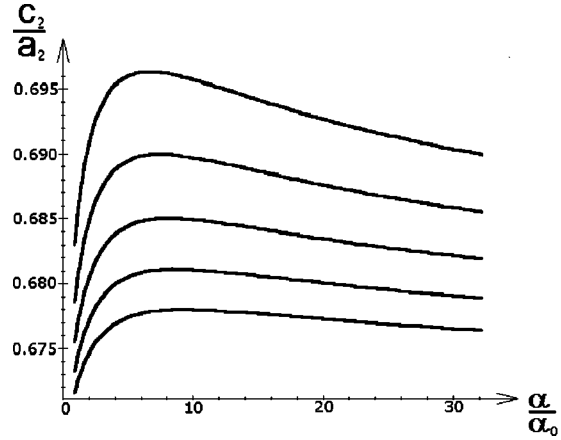

FIG. 2. Ratio c_2/a_2 of populations of metastable and stable states as a function of the dimensionless fluctuation rate α/α_0, where $\alpha_0=D/2a^2$. The parameters are: $L/a=1.1$, $\exp(-U_1/D)=0.2$, $\exp(-U_2/D)=0.3$, and (from top to bottom) $\exp(-\Delta/D)=0.2$, 0.25, 0.3, 0.35, and 0.40.

$$a_1=c_1=0, \quad a_3=-a_4\tanh(\beta L), \quad c_3=c_4\tanh(\beta L). \tag{12}$$

The matching conditions (3) result in

$$a_4=\frac{b_3\cosh(\beta a)+b_4\sinh(\beta a)}{\sinh(\beta a)-\tanh(\beta L)\cosh(\beta a)},$$

$$c_4=-\frac{b_3\cosh(\beta a)-b_4\sinh(\beta a)}{\sinh(\beta a)-\tanh(\beta L)\cosh(\beta a)}. \tag{13}$$

Since $a_1=c_1=0$, populations in regions 1 and 3, according to Eq. (9), are defined by the coefficients a_2 and c_2, respectively. After using (12) and (13), the five unknown coefficients, a_2, b_2, b_3, b_4, and c_2, have to be found from four equation (2) which, on using Eqs. (8)–(13) take the following form:

$$a_2-Bd_1=(b_2+e_1)\exp\left(\frac{U_1+\Delta}{D}\right),$$

$$a_2+Bd_1=(b_2-e_1)\exp\left(\frac{U_1}{D}\right),$$

$$c_2-Bd_2=(b_2-e_2)\exp\left(\frac{U_2+\Delta}{D}\right),$$

$$c_2+Bd_2=(b_2+e_2)\exp\left(\frac{U_2}{D}\right), \tag{14}$$

where

$$B=\coth[\beta(L-a)], \tag{15}$$

and

$$d_{1,2}=b_4\sinh(\beta a)\pm b_3\cosh(\beta a);$$

$$e_{1,2}=b_3\sinh(\beta a)\pm b_4\cosh(\beta a). \tag{16}$$

After simple but slightly tedious algebra one obtains from Eqs. (8)–(16),

$$\frac{c_2}{a_2}=\frac{u\{[16B_1B_2+2B_2\tanh(a\beta)]\}c^2+[16B_1B_2+4(B_1+B_2)\coth(a\beta)+4B_1\tanh(a\beta)+1]c+1+2B_2\tanh(a\beta)}{v\{[16B_1B_2+2B_1\tanh(a\beta)]\}c^2+[16B_1B_2+4(B_1+B_2)\coth(a\beta)+4B_2\tanh(a\beta)+1]c+1+2B_1\tanh(a\beta)}, \quad (17)$$

where

$$B_1=Bu, \quad B_2=Bv, \quad u=\exp\left(-\frac{U_1}{D}\right),$$
$$v=\exp\left(-\frac{U_2}{D}\right), \quad c=\exp\left(-\frac{\Delta}{D}\right). \quad (18)$$

One can check two limiting cases: (1) There are no fluctuations of the barrier ($\Delta=0$ or $\beta=0$); then the matching conditions (2),(3) at once result in Eq. (5), and Eq. (17) gives the correct result $c_2/a_2=\exp([U_1-U_2]/D)$; (2) potential barrier is symmetric ($U_1=U_2$); then, the matching conditions (2) and (3) lead to results coinciding with those obtained in Ref. [8] while Eq. (17) gives $c_2/a_2=1$, as it should be in the symmetric case.

Analysis of the general equation (17) leads to the following conclusions:

(a) One can easily find the limit values of c_2/a_2 for small and large Δ and β.

$$\frac{c_2}{a_2}=\frac{u}{v}\left[1+\frac{(v-u)B}{1+16B_1B_2+4(B_1+B_2)\coth(2\beta a)}\left(\frac{\Delta}{D}\right)^2\cdots\right] \quad \text{for } 0<\frac{\Delta}{D}<1,$$

$$\frac{c_2}{a_2}=\frac{u}{v}\left[1+\frac{2(v-u)B\tanh(\beta a)}{1+2B_1\tanh(\beta a)}\cdots\right] \quad \text{for } \frac{\Delta}{D}\to\infty, \quad (19)$$

$$\frac{c_2}{a_2}\approx\frac{u}{v}\left[1+\frac{(c-1)^2\left(\frac{L}{a}-1\right)(v-u)}{8uvc(c+1)+2c(u+v)\left(\frac{L}{a}-1\right)}(\beta a)^2+\cdots\right] \quad \text{for } 0<\beta a<1,$$

$$\frac{c_2}{a_2}=\frac{u}{v}\left[1+\frac{2(v-u)(c-1)^2}{2u(c+1)^2+8vc+(c+1)(1+16cuv)}+\cdots\right] \quad \text{for } \beta a\to\infty. \quad (20)$$

Since $v>u$ and $L>a$, Eqs. (19) and (20) show that the ratio c_2/a_2 of the mean populations in the left (metastable) and right (stable) wells increases with increasing fluctuation strength Δ or rate β. This corresponds to the stabilization of the metastable state.

(b) It turns out that c_2/a_2 increases monotonically with fluctuation strength Δ while the dependence on the fluctuation rate β is nonmonotonic. The nonmonotonic dependence of c_2/a_2 on β is of special interest within the framework of stochastic resonance phenomena [5]. The ratio of populations c_2/a_2 at metastable and stable state is shown in Fig. 2 as a function of the dimensionless fluctuation rate α/α_0 where $\alpha_0=D/2a^2$ for parameters $L/a=1.2$, $\exp(-U_1/D)=0.2$, $\exp(-U_2/D)=0.3$, and several values of fluctuation strengths.

Additional details concerning these maxima can be found from numerical analysis of Eq. (17). It turns out that the existence of maximum depends on the width of the potential well $(L-a)$. Maxima exist for $L/a\lesssim1.4$ and disappear at larger L/a. For the special case $L/a=2$ for which $B=\coth(\beta a)$, this result can be obtained analytically. The strong dependence on well width $L-a$ is not surprising since, as it was shown in Ref. [9] for the geometry of Fig. 1 with reflecting walls, the characteristic frequency of the system (Kramers rate) depends on L/a. It is remarkable that the heights of the potential barriers, a and b, scarcely influence the height and position of the maximum.

In conclusion, we have shown that an increase in population of a metastable state (''stabilization'' of the metastable state) can be achieved not only by the use of an external periodic field, but also by fluctuations of the barrier height.

[1] L. Landau and E. Lifshitz, *Mechanics*, 2nd ed. (Pergamon, Oxford, 1969), p. 95.
[2] J. Mills, Phys. Lett. A **133**, 295 (2000).
[3] M. Gitterman, J. Phys. A **34**, L355 (2001).
[4] V. Berdichevsky and M. Gitterman, Phys. Rev. E **59**, R9 (1999).
[5] L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[6] V. Berdichevsky and M. Gitterman, J. Phys. A **31**, 9773 (1998).
[7] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Science* (Springer, Berlin, 1983).
[8] J. Ankerhold and P. Pechukas, Phys. Rev. E **58**, 6968 (1998).
[9] V. Berdichevsky and M. Gitterman, J. Phys. A **29**, 1567 (1996).